Lagrangian dynamics of submanifolds. Relativistic mechanics

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Geometric formulation of Lagrangian relativistic mechanics in the terms of jets of one-dimensional submanifolds is generalized to Lagrangian theory of submanifolds of arbitrary dimension.

1 Introduction

Classical non-relativistic mechanics is adequately formulated as Lagrangian and Hamiltonian theory on a fibre bundle $Q \to \mathbb{R}$ over the time axis $\mathbb{R}$, where $\mathbb{R}$ is provided with the Cartesian coordinate t possessing the transition functions $t' = t$+const. [1, 5, 7, 8, 11]. A velocity space of non-relativistic mechanics is the first order jet manifold J^1Q of sections of $Q \to \mathbb{R}$. Lagrangians of non-relativistic mechanics are defined as densities on J^1Q. This formulation is extended to time-reparametrized non-relativistic mechanics subject to time-dependent transformations which are bundle automorphisms of $Q \to \mathbb{R}$ [5, 8].

Thus, one can think of non-relativistic mechanics as being particular classical field theory on fibre bundles over $X = \mathbb{R}$. However, an essential difference between non-relativistic mechanics and field theory on fibre bundles $Y \to X$, $\dim X > 1$, lies in the fact that connections on $Q \to \mathbb{R}$ always are flat. Therefore, they fail to be dynamic variables, but characterize non-relativistic reference frames.

In comparison with non-relativistic mechanics, relativistic mechanics admits transformations of the time depending on other variables, e.g., the Lorentz transformations in Special Relativity on a Minkowski space $Q = \mathbb{R}^4$. Therefore, a configuration space Q of relativistic mechanics has no preferable fibration $Q \to \mathbb{R}$, and its velocity space is the first order jet manifold J^1_1Q of one-dimensional submanifolds of a configuration space Q [5, 8, 12]. Fibres of the jet bundle $J^1_1Q \to Q$ are projective spaces, and one can think of them as being spaces of the three-velocities of a relativistic system. The four-velocities of a relativistic system are represented by elements of the tangent bundle TQ of a configuration space Q.

This work is devoted to generalization of the above mentioned formulation of relativistic mechanics to the case of submanifolds of arbitrary dimension.

Let us consider n-dimensional submanifolds of an m-dimensional smooth real manifold Z. The notion of jets of submanifolds [4, 6, 9] generalizes that of jets of sections of fibre bundles, which are particular jets of submanifolds (Section 2). Namely, a space of jets of submanifolds admits a cover by charts of jets of sections. Just as in relativistic mechanics, we restrict our consideration to first order jets of submanifolds which form a smooth manifold $J^1_n Z$. One however meets a problem how to develop Lagrangian formalism on a manifold $J^1_n Z$ because it is not a fibre bundle.

For this purpose, we associate to n-dimensional submanifolds of Z the sections of a trivial fibre bundle

$$\pi : Z_\Sigma = \Sigma \times Z \to \Sigma, \tag{1}$$

where Σ is some n-dimensional manifold. We obtain a relation between the elements of $J^1_n Z$ and the jets of sections of the fibre bundle (1)(Section 3). This relation fails to be one-to-one correspondence. The ambiguity contains, e.g., diffeomorphisms of Σ. Then Lagrangian formalism on a fibre bundle $Z_\Sigma \to \Sigma$ is developed in a standard way, but a Lagrangian is required to possess the gauge symmetry (20) which leads to the rather restrictive Noether identities (21) (Section 4).

If $n = 2$, this is the case, e.g., of the Nambu–Goto Lagrangian (22) of classical string theory (Example 1).

If $n = 1$, solving these Noether identities, we obtain a generic Lagrangian (43) of relativistic mechanics (Section 5).

These examples confirm the correctness of our description of Lagrangian dynamics of submanifolds of a manifold Z as that of sections of the fibre bundle Z_Σ (1).

2 Jets of submanifolds

Given an m-dimensional smooth real manifold Z, a k-order jet of n-dimensional submanifolds of Z at a point $z \in Z$ is defined as an equivalence class $j^k_z S$ of n-dimensional imbedded submanifolds of Z through z which are tangent to each other at z with order $k > 0$. Namely, two submanifolds

$$i_S : S \to Z, \qquad i_{S'} : S' \to Z$$

through a point $z \in Z$ belong to the same equivalence class $j^k_z S$ if and only if the images of the k-tangent morphisms

$$T^k i_S : T^k_z S \to T^k_z Z, \qquad T^k i_{S'} : T^k_z S' \to T^k_z Z$$

coincide with each other. The set

$$J^k_n Z = \bigcup_{z\in Z} j^k_z S$$

of k-order jets of submanifolds is a finite-dimensional real smooth manifold, called the k-order jet manifold of submanifolds [4, 6, 9].

Let $Y \to X$ be an m-dimensional fibre bundle over an n-dimensional base X and J^kY the k-order jet manifold of sections of $Y \to X$. Given an imbedding $\Phi : Y \to Z$, there is the natural injection

$$J^k\Phi : J^kY \to J^k_nZ, \qquad j^k_xs \to [\Phi \circ s]^k_{\Phi(s(x))},$$

where s are sections of $Y \to X$. This injection defines a chart on J^k_nZ. These charts provide a manifold atlas of J^k_nZ.

Let us restrict our consideration to first order jets of submanifolds. There is obvious one-to-one correspondence

$$\zeta : j^1_zS \to V_{j^1_zS} \subset T_zZ \tag{2}$$

between the jets j^1_zS at a point $z \in Z$ and the n-dimensional vector subspaces of the tangent space T_zZ of Z at z. It follows that J^1_nZ is a fibre bundle

$$\rho : J^1_nZ \to Z \tag{3}$$

with the structure group $GL(n, m-n; \mathbb{R})$ of linear transformations of a vector space $\mathbb{R}^m$ which preserve its subspace $\mathbb{R}^n$. The typical fibre of the fibre bundle (3) is a Grassmann manifold $GL(m; \mathbb{R})/GL(n, m-n; \mathbb{R})$. This fibre bundle is endowed with the following coordinate atlas.

Let $\{(U; z^A)\}$ be a coordinate atlas of Z. Let us provide Z with an atlas obtained by replacing every chart (U, z^A) of Z with the

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

charts on U which correspond to different partitions of (z^A) in collections of n and $m-n$ coordinates

$$(U; x^a, y^i), \qquad a = 1, \ldots, n, \qquad i = 1, \ldots, m-n. \tag{4}$$

The transition functions between the coordinate charts (4) associated with a coordinate chart (U, z^A) are reduced to exchange between coordinates x^a and y^i. Transition functions between arbitrary coordinate charts (4) take the form

$$x'^a = x'^a(x^b, y^k), \qquad y'^i = y'^i(x^b, y^k). \tag{5}$$

Let J^0_nZ denote a manifold Z provided with the coordinate atlas (4) – (5).

Given this atlas of $J^0_nZ = Z$, the first order jet manifold J^1_nZ is endowed with the coordinate charts

$$(\rho^{-1}(U) = U \times \mathbb{R}^{(m-n)n}; x^a, y^i, y^i_a), \tag{6}$$

possessing the following transition functions. With respect to the coordinates (6) on the jet manifold J^1_nZ and the induced fibre coordinates $(\dot x^a, \dot y^i)$ on the tangent bundle TZ, the above mentioned correspondence ζ (2) reads

$$\zeta : (y^i_a) \to \dot x^a(\partial_a + y^i_a(j^1_zS)\partial_i).$$

It implies the relations

$$y'^j_a = (\frac{\partial y'^j}{\partial y^k}y^k_b + \frac{\partial y'^j}{\partial x^b})(\frac{\partial x^b}{\partial y'^i}y'^i_a + \frac{\partial x^b}{\partial x'^a}), \tag{7}$$

$$(\frac{\partial x^b}{\partial y'^i}y'^i_a + \frac{\partial x^b}{\partial x'^a})(\frac{\partial x'^c}{\partial y^k}y^k_b + \frac{\partial x'^c}{\partial x^b}) = \delta^c_a, \tag{8}$$

which jet coordinates y^i_a must satisfy under coordinate transformations (5). Let us consider a non-degenerate $n \times n$ matrix M with the entries

$$M^c_b = (\frac{\partial x'^c}{\partial y^k}y^k_b + \frac{\partial x'^c}{\partial x^b}).$$

Then the relations (8) lead to the equalities

$$(\frac{\partial x^b}{\partial y'^i}y'^i_a + \frac{\partial x^b}{\partial x'^a}) = (M^{-1})^b_a.$$

Hence, we obtain the transformation law of first order jet coordinates

$$y'^j_a = (\frac{\partial y'^j}{\partial y^k}y^k_b + \frac{\partial y'^j}{\partial x^b})(M^{-1})^b_a. \tag{9}$$

In particular, if coordinate transition functions x'^a (5) are independent of coordinates y^k, the transformation law (9) comes to the familiar transformations of jets of sections.

3 The fibre bundle Z_Σ

Given a coordinate chart (6) of J^1_nZ, one can regard $\rho^{-1}(U) \subset J^1_nZ$ as the first order jet manifold J^1U of sections of a fibre bundle

$$\chi : U \ni (x^a, y^i) \to (x^a) \in \chi(U).$$

The graded differential algebra of exterior forms on $\rho^{-1}(U)$ is generated by horizontal forms dx^a and contact forms $dy^i - y^i_a dx^a$. Coordinate transformations (5) and (9) preserve the ideal of contact forms, but horizontal forms are not transformed into horizontal forms, unless coordinate transition functions x'^a (5) are independent of coordinates y^k. Therefore, one can develop first order Lagrangian formalism with a Lagrangian $L = \mathcal{L}d^nx$ on a coordinate chart $\rho^{-1}(U)$, but this Lagrangian fails to be globally defined on J^1_nZ.

In order to overcome this difficulty, let us consider the trivial fibre bundle $Z_\Sigma \to \Sigma$ (1) whose trivialization throughout holds fixed. This fibre bundle is provided with an atlas of coordinate charts

$$(U_\Sigma \times U; \sigma^\mu, x^a, y^i), \tag{10}$$

where $(U; x^a, y^i)$ are the above mentioned coordinate charts (4) of a manifold $J^0_n Z$. The coordinate charts (10) possess transition functions

$$\sigma'^\mu = \sigma^\mu(\sigma^\nu), \qquad x'^a = x'^a(x^b, y^k), \qquad y'^i = y'^i(x^b, y^k). \tag{11}$$

Let $J^1 Z_\Sigma$ be the first order jet manifold of the fibre bundle (1). Since the trivialization (1) is fixed, it is a vector bundle

$$\pi^1 : J^1 Z_\Sigma \to Z_\Sigma$$

isomorphic to the tensor product

$$J^1 Z_\Sigma = T^*\Sigma \underset{\Sigma\times Z}{\otimes} TZ \tag{12}$$

of the cotangent bundle $T^*\Sigma$ of Σ and the tangent bundle TZ of Z over Z_Σ.

Given the coordinate atlas (10) - (11) of Z_Σ, the jet manifold $J^1 Z_\Sigma$ is endowed with the coordinate charts

$$((\pi^1)^{-1}(U_\Sigma \times U) = U_\Sigma \times U \times \mathbb{R}^{mn}; \sigma^\mu, x^a, y^i, x^a_\mu, y^i_\mu), \tag{13}$$

possessing transition functions

$$x'^a_\mu = (\frac{\partial x'^a}{\partial y^k} y^k_\nu + \frac{\partial x'^a}{\partial x^b} x^b_\nu)\frac{\partial \sigma^\nu}{\partial \sigma'^\mu}, \quad y'^i_\mu = (\frac{\partial y'^i}{\partial y^k} y^k_\nu + \frac{\partial y'^i}{\partial x^b} x^b_\nu)\frac{\partial \sigma^\nu}{\partial \sigma'^\mu}. \tag{14}$$

Relative to the coordinates (13), the bundle isomorphism (12) takes the form

$$(x^a_\mu, y^i_\mu) \to d\sigma^\mu \otimes (x^a_\mu \partial_a + y^i_\mu \partial_i).$$

Obviously, a jet $(\sigma^\mu, x^a, y^i, x^a_\mu, y^i_\mu)$ of sections of the fibre bundle (1) defines some jet of n-dimensional submanifolds of a manifold $\{\sigma\} \times Z$ through a point $(x^a, y^i) \in Z$ if an $m \times n$ matrix with the entries (x^a_μ, y^i_μ) is of maximal rank n. This property is preserved under the coordinate transformations (14). An element of $J^1 Z_\Sigma$ is called regular if it possesses this property. Regular elements constitute an open subbundle of the jet bundle $J^1 Z_\Sigma \to Z_\Sigma$.

Since regular elements of $J^1 Z_\Sigma$ characterize first jets of n-dimensional submanifolds of Z, one hopes to describe the dynamics of these submanifolds of a manifold Z as that of sections of the fibre bundle (1). For this purpose, let us refine the relation between elements of the jet manifolds $J^1_n Z$ and $J^1 Z_\Sigma$.

Let us consider the manifold product $\Sigma\times J^1_nZ$. It is a fibre bundle over Z_Σ. Given the coordinate atlas (10) - (11) of Z_Σ, this product is endowed with the coordinate charts

$$(U_\Sigma\times\rho^{-1}(U)=U_\Sigma\times U\times\mathbb{R}^{(m-n)n};\sigma^\mu,x^a,y^i,y^i_a), \tag{15}$$

possessing the transition functions (9). Let us assign to an element $(\sigma^\mu,x^a,y^i,y^i_a)$ of the chart (15) the elements $(\sigma^\mu,x^a,y^i,x^a_\mu,y^i_\mu)$ of the chart (13) whose coordinates obey the relations

$$y^i_ax^a_\mu=y^i_\mu. \tag{16}$$

These elements make up an n^2-dimensional vector space. The relations (16) are maintained under the coordinate transformations (11) and the induced transformations of the charts (13) and (15) as follows:

$$\begin{aligned}
&y'^i_a x'^a_\mu=(\frac{\partial y'^i}{\partial y^k}y^k_c+\frac{\partial y'^i}{\partial x^c})(M^{-1})^c_a(\frac{\partial x'^a}{\partial y^k}y^k_\nu+\frac{\partial x'^a}{\partial x^b}x^b_\nu)\frac{\partial\sigma^\nu}{\partial\sigma'^\mu}=\\
&(\frac{\partial y'^i}{\partial y^k}y^k_c+\frac{\partial y'^i}{\partial x^c})(M^{-1})^c_a(\frac{\partial x'^a}{\partial y^k}y^k_b+\frac{\partial x'^a}{\partial x^b})x^b_\nu\frac{\partial\sigma^\nu}{\partial\sigma'^\mu}=\\
&(\frac{\partial y'^i}{\partial y^k}y^k_b+\frac{\partial y'^i}{\partial x^b})x^b_\nu\frac{\partial\sigma^\nu}{\partial\sigma'^\mu}=(\frac{\partial y'^i}{\partial y^k}y^k_\nu+\frac{\partial y'^i}{\partial x^b}x^b_\nu)\frac{\partial\sigma^\nu}{\partial\sigma'^\mu}=y'^i_\mu.
\end{aligned}$$

Thus, one can associate:

$$\zeta':(\sigma^\mu,x^a,y^i,y^i_a)\to\{(\sigma^\mu,x^a,y^i,x^a_\mu,y^i_\mu)\,|\,y^i_ax^a_\mu=y^i_\mu\},$$

to each element of a manifold $\Sigma\times J^1_nZ$ an n^2-dimensional vector space in a jet manifold J^1Z_Σ. This is a subspace of elements

$$x^a_\mu d\sigma^\mu\otimes(\partial_a+y^i_a\partial_i)$$

of a fibre of the tensor bundle (12) at a point (σ^μ,x^a,y^i). This subspace always contains regular elements, e.g., whose coordinates x^a_μ form a non-degenerate $n\times n$ matrix.

Conversely, given a regular element j^1_zs of J^1Z_Σ, there is a coordinate chart (13) such that coordinates x^a_μ of j^1_zs constitute a non-degenerate matrix, and j^1_zs defines a unique element of $\Sigma\times J^1_nZ$ by the relations

$$y^i_a=y^i_\mu(x^{-1})^\mu_a. \tag{17}$$

Thus, we have shown the following. Let (σ^μ,z^A) further be arbitrary coordinates on the product Z_Σ (1) and (σ^μ,z^A,z^A_μ) the corresponding coordinates on the jet manifold J^1Z_Σ. In these coordinates, an element of J^1Z_Σ is regular if an $m\times n$ matrix with the entries z^A_μ is of maximal rank n.

Theorem 1. (i) Any jet of submanifolds through a point $z \in Z$ defines some (but not unique) jet of sections of a fibre bundle Z_Σ (1) through a point $\sigma \times z$ for any $\sigma \in \Sigma$ in accordance with the relations (16).

(ii) Any regular element of $J^1 Z_\Sigma$ defines a unique element of a jet manifold $J^1_n Z$ by means of the relations (17). However, non-regular elements of $J^1 Z_\Sigma$ can correspond to different jets of submanifolds.

(iii) Two elements $(\sigma^\mu, z^A, z^A_\mu)$ and $(\sigma^\mu, z^A, z'^A_\mu)$ of $J^1 Z_\Sigma$ correspond to the same jet of submanifolds if

$$z'^A_\mu = M^\nu_\mu z^A_\nu,$$

where M is some matrix, e.g., it comes from a diffeomorphism of Σ.

4 Lagrangian formalism

Based on Theorem 1, we can describe the dynamics of n-dimensional submanifolds of a manifold Z as that of sections of the fibre bundle Z_Σ (1) for some n-dimensional manifold Σ.

Let

$$L = \mathcal{L}(z^A, z^A_\mu) d^n\sigma, \tag{18}$$

be a first order Lagrangian on a jet manifold $J^1 Z_\Sigma$. The corresponding Euler–Lagrange operator reads

$$\delta L = \mathcal{E}_A dz^A \wedge d^n\sigma, \qquad \mathcal{E}_A = \partial_A \mathcal{L} - d_\mu \partial^\mu_A \mathcal{L}. \tag{19}$$

It yields the Euler–Lagrange equations

$$\mathcal{E}_A = \partial_A \mathcal{L} - d_\mu \partial^\mu_A \mathcal{L} = 0.$$

In view of Theorem 1, it seems reasonable to require that, in order to describe jets of n-dimensional submanifolds of Z, the Lagrangian L (18) on $J^1 Z_\Sigma$ must be invariant under diffeomorphisms of a manifold Σ. To formulate this condition, it is sufficient to consider infinitesimal generators of one-parameter subgroups of these diffeomorphisms which are vector fields $u = u^\mu \partial_\mu$ on Σ. Since $Z_\Sigma \to \Sigma$ is a trivial bundle, such a vector field gives rise to a vector field $u = u^\mu \partial_\mu$ on Z_Σ. Its jet prolongation onto $J^1 Z_\Sigma$ reads

$$\begin{aligned} J^1 u &= u^\mu \partial_\mu - z^A_\nu \partial_\mu u^\nu \partial^\mu_A = u^\mu d_\mu + [-u^\nu z^A_\nu \partial_A - d_\mu(u^\nu z^A_\nu)\partial^\mu_A], \\ d_\mu &= \partial_\mu + z^A_\mu \partial_A + z^A_{\mu\nu} \partial^\nu_\mu + \cdots. \end{aligned} \tag{20}$$

One can regard it as a generalized vector field on $J^1 Z_\Sigma$ depending on parameter functions $u^\mu(\sigma^\nu)$, i.e., it is a gauge transformation [3, 4]. Let us require that $J^1 u$ (20) or, equivalently, its vertical part

$$u_V = -u^\nu z^A_\nu \partial_A - d_\mu(u^\nu z^A_\nu)\partial^\mu_A.$$

is a variational symmetry of the Lagrangian L (18). Then by virtue of the second Noether theorem, the Euler–Lagrange operator δL (19) obeys the irreducible Noether identities

$$z^A_\nu \mathcal{E}_A = 0. \tag{21}$$

One can think of these identities as being a condition which the Lagrangian L on $J^1 Z_\Sigma$ must satisfy in order to be a Lagrangian of submanifolds of Z. It is readily observed that this condition is rather restrictive.

Example 1. Let Z be a locally affine manifold, i.e., a toroidal cylinder $\mathbb{R}^{m-k} \times T^k$. Its tangent bundle can be provided with a constant non-degenerate fibre metric η_{AB}. Let Σ be a two-dimensional manifold. Let us consider the 2×2 matrix with the entries

$$h_{\mu\nu} = \eta_{AB} z^A_\mu z^B_\nu.$$

Then its determinant provides a Lagrangian

$$L = (\det h)^{1/2} d^2\sigma = ([\eta_{AB} z^A_1 z^B_1][\eta_{AB} z^A_2 z^B_2] - [\eta_{AB} z^A_1 z^B_2]^2)^{1/2} d^2\sigma \tag{22}$$

on the jet manifold $J^1 Z_\Sigma$ (12). This is the well known Nambu–Goto Lagrangian of classical string theory [10]. It satisfies the Noether identities (21).

5 Relativistic mechanics

As was mentioned above, if $n = 1$, we are in the case of relativistic mechanics. In this case, one can obtain a complete solution of the Noether identities (21) which provides a generic Lagrangian of relativistic mechanics.

Given an m-dimensional manifold Q coordinated by (q^λ), let us consider the jet manifold $J^1_1 Q$ of its one-dimensional submanifolds. It is treated as a velocity space of relativistic mechanics [5, 8, 12]. Let us provide $Q = J^0_1 Q$ with the coordinates (4):

$$(U; x^0 = q^0, y^i = q^i) = (U; q^\lambda). \tag{23}$$

Then the jet manifold $\rho : J^1_1 Q \to Q$ is endowed with the coordinates (6):

$$(\rho^{-1}(U); q^0, q^i, q^i_0) \tag{24}$$

possessing transition functions (5), (9):

$$q'^0 = q'^0(q^0, q^k), \qquad q'^0 = q'^0(q^0, q^k), \tag{25}$$

$$q'^i_0 = (\frac{\partial q'^i}{\partial q^j} q^j_0 + \frac{\partial q'^i}{\partial q^0})(\frac{\partial q'^0}{\partial q^j} q^j_0 + \frac{\partial q'^0}{\partial q^0})^{-1}. \tag{26}$$

A glance at the transformation law (26) shows that $J^1_1 Q \to Q$ is a fibre bundle in projective spaces.

Example 2. Let $Q = M^4 = \mathbb{R}^4$ be a Minkowski space whose Cartesian coordinates (q^λ), $\lambda = 0, 1, 2, 3$, are subject to the Lorentz transformations (25):

$$q'^0 = q^0 \mathrm{ch}\alpha - q^1 \mathrm{sh}\alpha, \quad q'^1 = -q^0 \mathrm{sh}\alpha + q^1 \mathrm{ch}\alpha, \quad q'^{2,3} = q^{2,3}. \tag{27}$$

Then q'^i (26) are exactly the Lorentz transformations

$$q'^1_0 = \frac{q^1_0 \mathrm{ch}\alpha - \mathrm{sh}\alpha}{-q^1_0 \mathrm{sh}\alpha + \mathrm{ch}\alpha} \qquad q'^{2,3}_0 = \frac{q^{2,3}_0}{-q^1_0 \mathrm{sh}\alpha + \mathrm{ch}\alpha}$$

of three-velocities in Special Relativity.

In view of Example 2, let us call a velocity space $J^1_1 Q$ of relativistic mechanics the space of three-velocities, though a dimension of Q need not equal $3+1$.

In order to develop Lagrangian formalism of relativistic mechanics, let us consider the trivial fibre bundle (1):

$$Q_R = \mathbb{R} \times Q \to \mathbb{R}, \tag{28}$$

whose base $\Sigma = \mathbb{R}$ is endowed with a global Cartesian coordinate τ. This fibre bundle is provided with an atlas of coordinate charts

$$(\mathbb{R} \times U; \tau, q^\lambda), \tag{29}$$

where $(U; q^0, q^i)$ are the coordinate charts (23) of a manifold $J^0_1 Q$. The coordinate charts (29) possess the transition functions (25). Let $J^1 Q_R$ be the first order jet manifold of the fibre bundle (28). Since the trivialization (28) is fixed, there is the canonical isomorphism (12) of $J^1 Q_R$ to the vertical tangent bundle

$$J^1 Q_R = V Q_R = \mathbb{R} \times TQ \tag{30}$$

of $Q_R \to \mathbb{R}$.

Given the coordinate atlas (29) of Q_R, a jet manifold $J^1 Q_R$ is endowed with the coordinate charts

$$((\pi^1)^{-1}(\mathbb{R} \times U) = \mathbb{R} \times U \times \mathbb{R}^m; \tau, q^\lambda, q^\lambda_\tau), \tag{31}$$

possessing transition functions

$$q'^\lambda_\tau = \frac{\partial q'^\lambda}{\partial q^\mu} q^\mu_\tau. \tag{32}$$

Relative to the coordinates (31), the isomorphism (30) takes the form

$$(\tau, q^\mu, q^\mu_\tau) \to (\tau, q^\mu, \dot{q}^\mu = q^\mu_\tau). \tag{33}$$

Example 3. Let $Q = M^4$ be a Minkowski space in Example 2 whose Cartesian coordinates (q^0, q^i) are subject to the Lorentz transformations (27). Then the corresponding transformations (32) take the form

$$q'^0_\tau = q^0_\tau \mathrm{ch}\alpha - q^1_\tau \mathrm{sh}\alpha, \quad q'^1_\tau = -q^0_\tau \mathrm{sh}\alpha + q^1_\tau \mathrm{ch}\alpha, \quad q'^{2,3}_\tau = q^{2,3}_\tau$$

of transformations of four-velocities in Special Relativity.

In view of Example 3, we agree to call fibre elements of $J^1Q_R \to Q_R$ the four-velocities though the dimension of Q need not equal 4. Due to the canonical isomorphism $q^\lambda_\tau \to \dot q^\lambda$ (30), by four-velocities also are meant the elements of the tangent bundle TQ, which is called the space of four-velocities.

In accordance with the terminology of Section 3, the non-zero jet (33) of sections of the fibre bundle (28) is regular, and it defines some jet of one-dimensional submanifolds of a manifold $\{\tau\} \times Q$ through a point $(q^0, q^i) \in Q$. Although this is not one-to-one correspondence, just as in Section 4, one can describe the dynamics of one-dimensional submanifolds of a manifold Q as that of sections of the fibre bundle (28).

Let us consider the manifold product $\mathbb{R} \times J^1_1 Q$. It is a fibre bundle over Q_R. Given the coordinate atlas (29) of Q_R, this product is endowed with the coordinate charts (15):

$$(U_R \times \rho^{-1}(U) = U_R \times U \times \mathbb{R}^{m-1}; \tau, q^0, q^i, q^i_0), \tag{34}$$

possessing transition functions (25) – (26). Let us assign to an element (τ, q^0, q^i, q^i_0) of the chart (34) the elements $(\tau, q^0, q^i, q^0_\tau, q^i_\tau)$ of the chart (31) whose coordinates obey the relations (16):

$$q^i_0 q^0_\tau = q^i_\tau. \tag{35}$$

These elements make up a one-dimensional vector space. The relations (35) are maintained under coordinate transformations (26) and (32). Thus, one can associate to each element of the manifold $\mathbb{R} \times J^1_1 Q$ a one-dimensional vector space

$$(\tau, q^0, q^i, q^i_0) \to \{(\tau, q^0, q^i, q^0_\tau, q^i_\tau) \,|\, q^i_0 q^0_\tau = q^i_\tau\}, \tag{36}$$

in a jet manifold J^1Q_R. This is a subspace of elements $q^0_\tau(\partial_0 + q^i_0\partial_i)$ of a fibre of the vertical tangent bundle (30) at a point (τ, q^0, q^i). Conversely, given a non-zero element (33) of J^1Q_R, there is a coordinate chart (31) such that this element defines a unique element of $\mathbb{R} \times J^1_1 Q$ by the relations (17):

$$q^i_0 = \frac{q^i_\tau}{q^0_\tau}. \tag{37}$$

Thus, we come to Theorem 1 for the case $n = 1$ as follows. Let (τ, q^λ) further be arbitrary coordinates on the product Q_R (28) and $(\tau, q^\lambda, q^\lambda_\tau)$ the corresponding coordinates on a jet manifold J^1Q_R.

Theorem 2. (i) Any jet of submanifolds through a point $q \in Q$ defines some (but not unique) jet of sections of the fibre bundle Q_R (28) through a point $\tau \times q$ for any $\tau \in \mathbb{R}$ in accordance with the relations (35).

(ii) Any non-zero element of $J^1 Q_R$ defines a unique element of the jet manifold $J^1_1 Q$ by means of the relations (37). However, non-zero elements of $J^1 Q_R$ can correspond to different jets of submanifolds.

(iii) Two elements $(\tau, q^\lambda, q^\lambda_\tau)$ and $(\tau, q^\lambda, q'^\lambda_\tau)$ of $J^1 Q_R$ correspond to the same jet of submanifolds if $q'^\lambda_\tau = r q^\lambda_\tau$, $r \in \mathbb{R} \setminus \{0\}$.

In the case of a Minkowski space $Q = M^4$ in Examples 2 and 3, the equalities (35) and (37) are the familiar relations between three- and four-velocities.

Let

$$L = \mathcal{L}(\tau, q^\lambda, q^\lambda_\tau) d\tau, \tag{38}$$

be a first order Lagrangian on a jet manifold $J^1 Q_R$. The corresponding Lagrange operator reads

$$\delta L = \mathcal{E}_\lambda dq^\lambda \wedge d\tau, \qquad \mathcal{E}_\lambda = \partial_\lambda \mathcal{L} - d_\tau \partial^\tau_\lambda \mathcal{L}. \tag{39}$$

Let us require that, in order to describe jets of one-dimensional submanifolds of Q, the Lagrangian L (38) on $J^1 Q_R$ possesses a gauge symmetry given by vector fields $u = u(\tau)\partial_\tau$ on Q_R or, equivalently, their vertical part

$$u_V = -u(\tau) q^\lambda_\tau \partial_\lambda, \tag{40}$$

which are generalized vector fields on Q_R. Then the variational derivatives of this Lagrangian obey the Noether identities (21):

$$q^\lambda_\tau \mathcal{E}_\lambda = 0. \tag{41}$$

We call such a Lagrangian the relativistic Lagrangian.

In order to obtain a generic form of a relativistic Lagrangian L, let us regard the Noether identities (41) as an equation for L. It admits the following solution. Let

$$\frac{1}{2N!} G_{\alpha_1 \ldots \alpha_{2N}}(q^\nu) dq^{\alpha_1} \vee \cdots \vee dq^{\alpha_{2N}}$$

be a symmetric tensor field on Q such that a function

$$G = G_{\alpha_1 \ldots \alpha_{2N}}(q^\nu) \dot{q}^{\alpha_1} \cdots \dot{q}^{\alpha_{2N}} \tag{42}$$

is positive ($G > 0$) everywhere on $TQ \setminus \widehat{0}(Q)$, where $\widehat{0}(Q)$ is the global zero section of $TQ \to Q$. Let $A = A_\mu(q^\nu) dq^\mu$ be a one-form on Q. Given the pull-back of G and A onto $J^1 Q_R$ due to the canonical isomorphism (30), we define a Lagrangian

$$L = (G^{1/2N} + q^\mu_\tau A_\mu) d\tau, \qquad G = G_{\alpha_1 \ldots \alpha_{2N}} q^{\alpha_1}_\tau \cdots q^{\alpha_{2N}}_\tau, \tag{43}$$

on $J^1Q_R \setminus (\mathbb{R}\times \widehat{0}(Q))$. The corresponding Lagrange equations read

$$\mathcal{E}_\lambda = \frac{\partial_\lambda G}{2NG^{1-1/2N}} - d_\tau\left(\frac{\partial^\tau_\lambda G}{2NG^{1-1/2N}}\right) + F_{\lambda\mu}q^\mu_\tau = \qquad (44)$$

$$E_\beta[\delta^\beta_\lambda - q^\beta_\tau G_{\lambda\nu_2\ldots\nu_{2N}}q^{\nu_2}_\tau\cdots q^{\nu_{2N}}_\tau G^{-1}]G^{1/2N-1} = 0,$$

$$E_\beta = \left(\frac{\partial_\beta G_{\mu\alpha_2\ldots\alpha_{2N}}}{2N} - \partial_\mu G_{\beta\alpha_2\ldots\alpha_{2N}}\right)q^\mu_\tau q^{\alpha_2}_\tau\cdots q^{\alpha_{2N}}_\tau - \qquad (45)$$

$$(2N-1)G_{\beta\mu\alpha_3\ldots\alpha_{2N}}q^\mu_{\tau\tau}q^{\alpha_3}_\tau\cdots q^{\alpha_{2N}}_\tau + G^{1-1/2N}F_{\beta\mu}q^\mu_\tau,$$

$$F_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda.$$

It is readily observed that the variational derivatives $\mathcal{E}_\lambda$ (44) satisfy the Noether identities (41). Moreover, any relativistic Lagrangian obeying the Noether identity (41) is of type (43).

A glance at the Lagrange equations (44) shows that they hold if

$$E_\beta = \Phi G_{\beta\nu_2\ldots\nu_{2N}}q^{\nu_2}_\tau\cdots q^{\nu_{2N}}_\tau G^{-1}, \qquad (46)$$

where Φ is some function on J^1Q_R. In particular, we consider the equations

$$E_\beta = 0. \qquad (47)$$

Because of the Noether identities (41), the system of equations (44) is underdetermined. To overcome this difficulty, one can complete it with some additional equation. Given the function G (43), let us choose the condition

$$G = 1. \qquad (48)$$

Being positive, the function G (43) possesses a nowhere vanishing differential. Therefore, its level surface W_G defined by the condition (48) is a submanifold of J^1Q_R.

Our choice of the equations (47) and the condition (48) is motivated by the following facts.

Lemma 3. Any solution of the Lagrange equations (44) living in the submanifold W_G is a solution of the equation (47).

Proof. A solution of the Lagrange equations (44) living in the submanifold W_G obeys the system of equations

$$\mathcal{E}_\lambda = 0, \qquad G = 1. \qquad (49)$$

Therefore, it satisfies the equality

$$d_\tau G = 0. \qquad (50)$$

Then a glance at the expression (44) shows that the equations (49) are equivalent to the equations

$$\begin{aligned}E_\lambda = &\left(\frac{\partial_\lambda G_{\mu\alpha_2\ldots\alpha_{2N}}}{2N} - \partial_\mu G_{\lambda\alpha_2\ldots\alpha_{2N}}\right) q^\mu_\tau q^{\alpha_2}_\tau \cdots q^{\alpha_{2N}}_\tau - \\ &(2N-1) G_{\beta\mu\alpha_3\ldots\alpha_{2N}} q^\mu_{\tau\tau} q^{\alpha_3}_\tau \cdots q^{\alpha_{2N}}_\tau + F_{\beta\mu} q^\mu_\tau = 0, \\ G = &G_{\alpha_1\ldots\alpha_{2N}} q^{\alpha_1}_\tau \cdots q^{\alpha_{2N}}_\tau = 1.\end{aligned} \tag{51}$$

□

Lemma 4. Solutions of the equations (47) do not leave the submanifold W_G (48).

Proof. Since

$$d_\tau G = -\frac{2N}{2N-1} q^\beta_\tau E_\beta,$$

any solution of the equations (47) intersecting the submanifold W_G (48) obeys the equality (50) and, consequently, lives in W_G. □

The system of equations (51) is called the relativistic equation. Its components E_λ (45) are not independent, but obeys the relation

$$q^\beta_\tau E_\beta = -\frac{2N-1}{2N} d_\tau G = 0, \qquad G = 1,$$

similar to the Noether identities (41). The condition (48) is called the relativistic constraint.

Though the system of equations (44) for sections of a fibre bundle $Q_R \to \mathbb{R}$ is underdetermined, it is determined if, given a coordinate chart $(U; q^0, q^i)$ (23) of Q and the corresponding coordinate chart (29) of Q_R, we rewrite it in the terms of three-velocities q^i_0 (37) as equations for sections of a fibre bundle $U \to \chi(U) \subset \mathbb{R}$.

Let us denote

$$\overline{G}(q^\lambda, q^i_0) = (q^0_\tau)^{-2N} G(q^\lambda, q^\lambda_\tau), \qquad q^0_\tau \neq 0. \tag{52}$$

Then we have

$$\mathcal{E}_i = q^0_\tau \left[\frac{\partial_i \overline{G}}{2N\overline{G}^{1-1/2N}} - (q^0_\tau)^{-1} d_\tau \left(\frac{\partial^0_i \overline{G}}{2N\overline{G}^{1-1/2N}}\right) + F_{ij} q^j_0 + F_{i0}\right].$$

Let us consider a solution $\{s^\lambda(\tau)\}$ of the equations (44) such that $\partial_\tau s^0$ does not vanish and there exists an inverse function $\tau(q^0)$. Then this solution can be represented by sections

$$s^i(\tau) = (\overline{s}^i \circ s^0)(\tau) \tag{53}$$

of the composite bundle

$$\mathbb{R}\times U \to \mathbb{R}\times \pi(U) \to \mathbb{R}$$

where $\overline{s}^i(q^0) = s^i(\tau(q^0))$ are sections of $U \to \chi(U)$ and $s^0(\tau)$ are sections of $\mathbb{R}\times\pi(U) \to \mathbb{R}$. Restricted to such solutions, the equations (44) are equivalent to the equations

$$\begin{aligned}
\overline{\mathcal{E}}_i &= \frac{\partial_i\overline{G}}{2N\overline{G}^{1-1/2N}} - d_0\left(\frac{\partial_i^0\overline{G}}{2N\overline{G}^{1-1/2N}}\right) + F_{ij}q_0^j + F_{i0} = 0, \\
\overline{\mathcal{E}}_0 &= -q_0^i\overline{\mathcal{E}}_i.
\end{aligned} \tag{54}$$

for sections $\overline{s}^i(q^0)$ of a fibre bundle $U \to \chi(U)$.

It is readily observed that the equations (54) are the Lagrange equation of the Lagrangian

$$\overline{L} = (\overline{G}^{1/2N} + q_0^iA_i + A_0)dq^0 \tag{55}$$

on the jet manifold J^1U of a fibre bundle $U \to \chi(U)$.

It should be emphasized that, both the equations (54) and the Lagrangian (55) are defined only on a coordinate chart (23) of Q since they are not maintained under the transition functions (25) – (26).

A solution $\overline{s}^i(q^0)$ of the equations (54) defines a solution $s^\lambda(\tau)$ (53) of the equations (44) up to an arbitrary function $s^0(\tau)$. The relativistic constraint (48) enables one to overcome this ambiguity as follows.

Let us assume that, restricted to the coordinate chart $(U; q^0, q^i)$ (23) of Q, the relativistic constraint (48) has no solution $q_\tau^0 = 0$. Then it is brought into the form

$$(q_\tau^0)^{2N}\overline{G}(q^\lambda, q_0^i) = 1, \tag{56}$$

where $\overline{G}$ is the function (52). With the condition (56), every three-velocity (q_0^i) defines a unique pair of four-velocities

$$q_\tau^0 = \pm(\overline{G}(q^\lambda, q_0^i))^{1/2N}, \qquad q_\tau^i = q_\tau^0 q_0^i. \tag{57}$$

Accordingly, any solution $\overline{s}^i(q^0)$ of the equations (54) leads to solutions

$$\tau(q^0) = \pm\int(\overline{G}(q^0, \overline{s}^i(q^0), \partial_0\overline{s}^i(q_0))^{-1/2N}dq^0, \quad s^i(\tau) = s^0(\tau)(\partial_i\overline{s}^i)(s^0(\tau))$$

of the equations (49) and, equivalently, the relativistic equations (51).

Example 4. Let $Q = M^4$ be a Minkowski space provided with the Minkowski metric $\eta_{\mu\nu}$ of signature $(+,- - -)$. This is the case of Special Relativity. Let $\mathcal{A} = \mathcal{A}_\lambda dq^\lambda$ be a one-form on Q. Then

$$L = [m(\eta_{\mu\nu}q_\tau^\mu q_\tau^\nu)^{1/2} + e\mathcal{A}_\mu q_\tau^\mu]d\tau, \qquad m, e \in \mathbb{R}, \tag{58}$$

is a relativistic Lagrangian on J^1Q_R which satisfies the Noether identity (41). The corresponding relativistic equation (51) reads

$$m\eta_{\mu\nu}q^\nu_{\tau\tau} - eF_{\mu\nu}q^\nu_\tau = 0, \tag{59}$$

$$\eta_{\mu\nu}q^\mu_\tau q^\nu_\tau = 1. \tag{60}$$

This describes a relativistic massive charge in the presence of an electromagnetic or gauge potential $\mathcal{A}$. It follows from the relativistic constraint (60) that $(q^0_\tau)^2 \geq 1$. Therefore, passing to three-velocities, we obtain the Lagrangian (55):

$$\overline{L} = \left[m(1 - \sum_i (q^i_0)^2)^{1/2} + e(\mathcal{A}_i q^i_0 + \mathcal{A}_0)\right] dq^0,$$

and the Lagrange equations (54):

$$d_0\left(\frac{mq^i_0}{(1 - \sum_i (q^i_0)^2)^{1/2}}\right) + e(F_{ij}q^j_0 + F_{i0}) = 0.$$

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